Designs for Grounded and Embodied Mathematical Learning

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Abstract: Findings synthesized across five empirical, laboratory- and classroom-based studies of high school and college students engaged in geometric reasoning and proof production during single and multi-session investigations (346 participants overall) are presented. The findings converge on several design principles for embodied mathematical thinking and learning. *The Hidden Village* is presented as an instance of an embodied learning environment that uses the narrative context of a visual novel to instantiate these design principles and investigate their influences on mathematical thinking and learning, and to inform a broader theoretical framework of grounded and embodied learning. The interplay of theoretical, empirical, and design considerations of grounded and embodied learning reframes learning, transfer, and assessment, offering promising new pathways for an emerging class of learning environments.

Grounded and embodied learning

Grounded and embodied learning (GEL) offers a framework for understanding and designing for meaning making based on body-based processes. These GEL processes include actions, gestures, operative speech, and collaborative movements, as well as processes for supporting simulated action and imagination. Evidence is mounting that GEL is also activated during interventions, such as when learners are prompted to perform directed actions or are prompted to engage in simulated actions. This paper summarizes a set of interwoven empirical, theoretical, and design advancements for mathematical reasoning and their implications for teaching, learning, learning environment design, and assessment.

The **theoretical contributions** offer hypothesis-driven inquiry into the nature of learning and meaning making through body-based experiences, and principled guidance for the design of learning technology and educational experiences. **Empirical** contributions span several laboratory- and classroom-based studies of high school and college students and high school teachers; in all, some 346 participants who engaged in 5 single and multi-session investigations. These investigations provide evidence of the role of embodied processes for meaning making in support of mathematical reasoning and theoretically guided design principles to promote learning.

Design advancements to technologies for embodied thinking and learning serve as essential instantiations of theoretically derived hypotheses, as well as vital spaces in which embodied behaviors are elicited and data collected. Empirical findings and theoretical progress inform improvements in learning environment design that then make possible further theoretical and empirical advancements. In the final section, we explore how the interplay of evidence, theory, and design offer pathways for progress on complex issues facing GEL, such as generalizing research, bridging research and practice, and the emergence of designs and design principles.

Theoretical Frame: Grounded and Embodied Mathematical Cognition

Assumptions About Cognition

Memories, experiences, and meaning are constructed through the continuous *interactions* among environmental, social, cognitive, motoric, and perceptual processes of a highly dynamic, self-regulating organism, via a *perception-action loop* (Neisser, 1976; Varela et al., 1991). The cognitive system is a *predictive architecture*, continually anticipating responses to sensorial input from the environment and from within (Clark, 2013; Glenberg, 1997). People engage in cognitive processes that *extend* beyond the individual actor. Task-relevant knowledge is *embedded* in the situations in which activity unfolds and is *distributed* across actors, objects, and space (Clark & Chalmers, 1998). People can regulate distributed resources through cognitive *offloading* (Hutchins, 1995; Wilson, 2002). Finally, to be meaningful, new ideas, abstractions, and symbolic representations must be *grounded* in one's lived, sensorimotor experiences (Harnad, 1990; Glenberg et al., 2004).

Several scholars have synthesized evidence-based design principles for fostering embodied learning (Abrahamson et al., 2020; Lindgren & Johnson-Glenberg, 2013; Johnson-Glenberg et al., 2014; Skulmowski & Rey, 2018), with considerable overlap on these considerations: Motoric engagement; cognitive relevance of actions (aka "action-concept congruency"); and presence, including virtual, sensorial, and social immersion.

Integrating Linguistic and Body-Based Processes

Mathematical cognition emerges from the interplay of symbolic, verbal, and sensorimotor knowledge and processes. Symbol systems are important disciplinary representations that codify formal knowledge, yet their arbitrary, abstract, and amodal (i.e., ungrounded) nature poses challenges for learners (Nathan, 2012). Oral language serves as a source for consciously grounding symbol systems, expressing meaning, and representing abstract relations (Koedinger, Alibali & Nathan, 2008). Sensorimotor processes offer a complement, through nonverbal and nonsymbolic forms of knowing based on one's lived experiences. Through the cognitive mechanism of conceptual metaphor, bodily processes can be the basis for grounding foundational mathematical concepts, such as number, as well as the means to *link* to more advanced topics (Lakoff & Núñez, 2000).

Gestures are spontaneous acts of the body—most typically the hands and arms—used during mathematical learning and teaching that integrate naturally with public speech (social communication) and inner speech (verbalizable thinking) to make direct indexical reference to the world (deixis), depict worldly objects (iconics), and represent abstract relations (metaphorics) (Alibali & Nathan, 2012). Gesture has three special qualities relevant for embodied mathematical cognition. First, it can manifest simulated actions (Hostetter & Alibali, 2008). As an example, one does not merely imagine *100* as a symbol, but as a location along a mental number line, with a corresponding gesture that may sweep to the right of the body's central axis. Second, gestures can add abstract as well as sensorimotor information from actions into the cognitive encoding of a task. In effect, gestures help to *schematize* the most important qualities of task-relevant actions—physical and simulated actions—that can facilitate the formation and generalization of task-relevant cognitive processes for future use (Kita et al., 2017). As such, gestures "provide a bridge between concrete actions and more abstract representation" (Goldin-Meadow & Beilock, 2010, p. 672). Third, gestures express multimodal knowledge that cannot be verbalized, and which may be unconscious or as yet linguistically and symbolically unformed.

Action-Cognition Transduction: How Actions Produce Ideas

Cognitive processes have historically been framed using computer-based metaphors. Within this metaphor, a "central processing unit" governs the behaviors of "peripherals" such as perceptual systems, and output devices in the form of actuators such as our hands, that act upon the world (Eisenberg, 2002). We readily accept that our thoughts can (often) control our actions. *Action–cognition transduction* (ACT) explains how, in reciprocal fashion, body movement can induce mental states and actions through the complex coupling of feedforward and feedback interactions (Nathan, 2017). This alternative cognitive architecture places the function of peripherals as also central to cognition, through means such as gestures, eye gaze patterns, epistemic actions, and cognitive offloading (Kirsch & Maglio, 1994; Wilson, 2002). Actions, in this view, literally change our minds.

Geometric Reasoning and Mathematical Proof Production

Embodied approaches to thinking and learning confront some unique challenges regarding mathematical proof. Proof is a central disciplinary practice among mathematicians, serving as the primary method for testing claims, constructing knowledge, and disseminating research (Lakatos, 2015). Geometry is the primary scholastic content area for k-12 students learning to produce mathematically valid proofs and is used to explore properties of space and shape (Lehrer & Chazan, 2012; NCTM, 2000; Stylianides, 2007). As a topic of advanced mathematics, it is steeped in abstractions and formalizations to make universal claims that go well beyond personal experiences. Formulating proofs does not rest on well-rehearsed procedures. It requires the development of mathematical intuition along with logical, generalizable, and goal-directed, operational thought (Harel & Sowder, 2005).

Despite its importance, educational systems struggle to cultivate proof practices (Dreyfus, 1999). As examples, students rely on tautological claims without cultivating true insights about the phenomena of interest and tend to overgeneralize claims derived from specific examples (e.g., Healy & Hoyles, 2000). This prompted our inquiry about ways to promote GEL of proof practices.

Theoretically Guided Research Questions

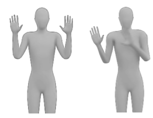
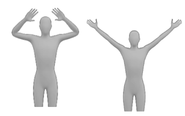
We report on the findings from several empirical studies designed to explore the following questions: (1) What are the emerging cognitive principles for GEL of geometry proof? (2) How can these principles inform the design of computer-based embodied learning experiences?

Empirical Support for GEL of Mathematical Reasoning

Dynamic Gestures Enact Simulated Actions to Facilitate Geometric Reasoning

This first study (Nathan et al., 2020) asked: *Is geometric reasoning associated with participants' gesture production?* Secondarily, we wanted to know: *Does the strength of the relationship between geometric reasoning* *and gesture production depend on participants' mathematical expertise?* We analyzed video recordings of 46 mathematics experts and 44 non-experts recruited from a large US midwestern university who evaluated the truth of conjectures about geometric properties (Table 1). We used logistic regression for binary outcomes (0/1) to model accuracy of intuition (snap judgments), insight (gist), and proof. A valid proof had to show reasoning that is logical, operational, and generalizable across the class of mathematical objects under consideration.

Table 1
Truth value, insights, and proof for example mathematical conjectures.

Conjecture Text	Truth	Insight	Proof	Cognitively Relevant Directed Actions
The area of a parallelogram is the same as the area of a rectangle with the same length and height	True	(1) States a parallelogram is a rectangle tilted or pushed over. (2) States area of a parallelogram and a rectangle have the same formula.	(1) Shows cutting off a triangle from the parallelogram, or rearranging the area makes them congruent. (2) State all rectangles are parallelograms thus the formula for area is the same.	
Given that you know the measure of all three angles of a triangle, there is only one unique triangle that can be formed with these three angle measurements	False	States similar triangles or infinite/many triangles	(1) Gives specific counterexample (2) Visually shows scaling or discusses scaling and similar triangles	

Analyses provide empirical support for claims that geometry proof production is an embodied activity, even when controlling for math expertise, language use, and spatial ability. We coded for *dynamic depictive gestures* that portray generalizable properties of shape and space through enactment of transformational operations (e.g., dilation, skewing; Garcia & Infante, 2012; Nathan & Walkington, 2017). *Dynamic depictive gestures* (Figure 1) are contrasted with non-dynamic depictive gestures that may trace or merely refer to mathematical objects and their transformations. Non-dynamic depictive gestures were associated with correct mathematical intuitions (p = .004). Dynamic depictive gestures were more likely to be associated with correct insights (p = 0.042) and mathematically valid proofs ($p < .0001$) than when no such gestures were made. Furthermore, experts significantly out-performed non-experts and were more likely to produce dynamic gestures (79%) than non-experts (52%).

The benefits of dynamic gestures were replicated among undergraduates from a large, southwestern university (N = 108). There, production of dynamic gestures was associated with mathematical insights (d = 0.45, $p < .05$) and mathematically valid proof (d = 0.74, $p < 0.001$), but production of any gesture was not (Walkington et al., 2019). Dynamic gestures may allow participants to explore generalized properties of shapes and space, thus supporting their geometric reasoning. This suggests that gestures, as simulated actions, provide an alternative account of mathematical reasoning than that of symbolic cognition. This raises questions of whether and how to design learning experiences that elicit gestures in service of mathematical thinking and learning.

Directed Actions Foster Reasoning and Performance Benefits Through Video Game Play: The Importance of Cognitive Relevance

To explore how eliciting gestures impacts mathematical reasoning, we developed *The Hidden Village* (THV), an embodied video game that uses the *visual novel* genre (Cavallaro, 2009). We explored to questions: *Can we use directed actions to elicit dynamic gestures that will enhance geometry proof performance?* And *Does the cognitive relevance of the directed actions used in the game matter?*

A player mimics movements of in-game "villagers" doing activities such as dance and play (Figure 2). THV uses 3D motion-capture technology via the Kinect™ sensor array to track player motion in real time to determine if the player has performed the intended directed action. *Directed actions* are carefully chosen movements curated from prior sessions of successful mathematical reasoning. Once the directed actions are repeatedly matched, the player reads a geometry conjecture (Table 1), decides if it always true or ever false, provides an explanation in speech and gesture (which is recorded), and then chooses a multiple-choice response.

Completing each conjecture cycle earns a knowledge token and reveals a bit more of the map of THV.

In this experiment (Walkington et al., 2021), 85 high school students (65 female) from a program for soon-to-be first-generation college students were recruited. Forty-eight identified as Hispanic, 26 as African American, 5 as Asian, 3 as Caucasian, 2 as other race/ethnicity, and 1 had race/ethnicity data missing. Each student played eight conjectures. Cognitive relevance of the directed actions to each conjecture was manipulated within-subjects (4 relevant, 4 irrelevant). For example, a conjecture poses "The area of a parallelogram is the same as the area of a rectangle with the same length and height." The cognitively relevant directed actions in Table 1 schematize the conjecture's mathematical relations (Nathan & Walkington, 2017). The *irrelevant* directed actions are recycled actions from other conjectures to vary relevance while controlling for physical complexity. Conjecture order was counterbalanced using an 8-item Latin square factorial design. Participants were not informed that any of their directed actions were relevant to the conjectures they were being asked to prove.

Data were analyzed using mixed effects logistic regression models with repeated observations of students solving conjectures nested within student. Participant ID was made a random effect with an additional random effect for conjecture. Condition (relevant v. irrelevant) was not a reliable predictor of gesture usage on its own. Still, when participants were cued with relevant actions, they showed superior proof performance compared to times when cued to make irrelevant actions, *provided* they made any gestures ($d = 1.13$, $p < 0.01$), or, more strongly, when they made dynamic depictive gestures ($d = 1.56$, $p < 0.01$). Similar advantages for cognitive relevance were observed for performance on mathematical insight ($d = 1.02$, $p <.01$) and intuition ($d = 1.00$, $p <.01$), *so long as* students produced dynamic gestures during their explanations.

The results suggest that cognitively relevant directed actions contribute to superior mathematical reasoning. This beneficial influence is not direct, however, but appears to be *moderated* by the role of dynamic gestures during students' multimodal explanations. This may be because explanatory gestures help to schematize relevant mathematical information (Kita et al., 2017), thus enabling students to make better use of the conceptual information they obtain via action-cognition transduction when they perform directed actions that are cognitively relevant. Comparable actions that are not cognitively relevant offer fewer benefits to mathematical reasoning.

Collaborative Gestures Enhance Geometric Reasoning Via Extended Cognition

Collaboration offers another potential way to increase students' production of dynamic gestures during mathematical reasoning, supporting enhanced performance. When high school students (N = 51) from a school district in the Southwestern US were observed playing *The Hidden Village* in groups, they operated as a form of distributed system, with mathematical thinking extended across multiple members of the group in speech as well as gestures. Walkington and colleagues (2019, 2021) documented several types of *collaborative gestures,* defined as "gestures that are physically and gesturally taken up by multiple learners, holding a meaning that is explicitly dependent upon the gestures of interactional partners." Forms of collaborative gesture that students exhibited included (Figure 3): echoing (55.5%), as when a gesture was repeated; mirroring (24.4%), as when an observer matches the speaker's gestures at the same time; alternating (20%), as when an observer proposes an alternate gesture to advance the group's mathematical line of thinking without copying (as in echoing); and joint gestures (15.5%), as when multiple learners use their hands conjointly to form a single mathematical object or idea.

They found that gestures--especially collaborative gesture production--contributed significantly to student performance. When group members did not gesture, students generated the correct insight 12% of the time. In comparison, the correct insight was produced more often with non-collaborative gestures (39%) and far more often with collaborative gestures (72%). Extending body-based resources across multiple participants, as well as other resources, reveals additional ways embodiment facilitates cognition.

Embodied Knowledge Transfer and the Travel of Ideas

One of the hallmarks of learning is to foster transfer, the conditions in which prior experiences facilitate learning and extend its reach to novel, but related areas of performance. Goldstone, Landy, and Son (2008) demonstrate how perceptual learning transfers "not through acquiring and applying symbolic formalisms but rather through modifying automatically perceived similarities between scenarios by training one's perceptual interpretations" (p. 329). In this small-scale study (Kirankumar et al., 2021), 12 students in a Title I high school played THV in small groups during Day 1. On Day 2 they used the THV-Conjecture Editor (Figure 4) to co-create new game content for their classmates. In so doing, they had to create cognitively relevant directed actions that would foster mathematical learning among their peers for Day 3, but without telling them—only through game play.

Group A chose to create the *ABC Reflection* conjecture (which is false)*: Given three points A, B, and C, and their reflected images about a line, A', B', and C', then ∠ ABC and ∠ A'B'C' are not equal.* Within-group analyses of their collaborative co-design (Day 2) showed that students regularly produced dynamic depictive gestures while investigating which directed actions would assist new players in forming the proper geometric relations relevant to proving their conjecture. Between-group analyses (Day 3) showed that students in Groups B and D learned some insights and proof elements by playing the *ABC Reflection* conjecture from Group A.

The spontaneous gestures produced by students in Group D following Group A's directed actions demonstrated that game play provided an embodied conceptualization of reflection. The gesture made by a student in Group B reveals an understanding of reflection based on the directed actions performed by their teammate. This is demonstrated using a hand flip rather than whole arm/torso flip, showing how the student tracked the key invariant relation (angular measure), mapping that relation across contexts and bodies through the student's own body movement. The action (hand flip) performed in the new context can activate common cognitive states through transduction, which highlights the invariant relation in its new setting. This highlight helps establish the mapping by preserving what must remain invariant as it travels across contexts and physical settings.

These two cases provide preliminary insights into how ideas can "travel" via embodied engagement through game content creation and game play. It illustrates how "transfer is an embedded process, situated in a particular physical and sociocultural learning context" (Nathan & Alibali, 2021, p. 52).

Discussion

With these empirical findings in mind, we revisit the research questions. For the first question, we highlight several emerging design principles for GEL of geometry proof. One principle is the apparent cognitive benefits of *integrating linguistic and body-based processes*: Gestures are multimodal linguistic forms that both express and influence students' mathematical reasoning in ways that are both nonverbal (intuition) and verbal (insight and proofs), even when controlling for spatial ability and prior math education. Dynamic gestures in particular are shown to support the simulated actions (Hostetter & Alibali, 2008, 2019) needed for hypothesizing and generalizing universal truths about space and shape, which is central to geometry proof practices.

A second principle is that the *cognitive relevance of directed actions* is critical for designing effective interventions to enhance the quality of students' proofs. This builds on the notion that decentralizes cognition by recognizing that cognition and goal-directed actions mutually and reciprocally influence one another through transduction (Nathan, 2017). The mapping of concepts to relevant actions is, however, not a simple one-to-one relationship, which illustrates its complexity while also inviting tremendous variety for the designer. This design constraint appears to rest on the degree to which learners use explanatory gesture (again, integrating linguistic and bodily processes) to schematize the key mathematical relationships through movement (Walkington et al., 2021).

Creating collaborative contexts is a third principle shown to foster the kinds of gestures that are beneficial to mathematical reasoning (Walkington et al., 2019). In addition, the collaborative interactions that arise in these contexts support the formation of distributed knowledge exhibited by students' collaborative gestures, and the tightly coupled interactions that foster shared meaning. The final principle is *designing for embodied transfer by supporting production of prior actions*. Transfer emerges from the embodied nature of learning and the natural tendency of learners to predict the world and build cohesion across their learning experiences (Nathan & Alibali, 2021). In this sense, transfer is the default mode for engaged learners. Recurrent action sequences can bridge the travel of ideas across contexts, and educational environments need to be designed with this in mind. It also underscores how transfer is inherently embedded in socially mediated interactions among learners with teachers, peers, and technologies, allowing ideas to "travel" among groups and across contexts.

The second research question asks how these principles inform the design of computer-based embodied learning experiences. We explored these design principles in the context of *The Hidden Village* (THV), a visual novel used to engage players in embodied mathematical reasoning. THV is designed to integrate linguistic- and body-based processes during learning and performance assessment. It does this by eliciting from players the movements that were curated from prior sessions of successful mathematical reasoning. Following this, THV collects data on players' nonverbal intuitions (i.e., snap judgments) about the truth of the mathematical conjectures of interest. It then elicits these movements during players' multimodal explanations and justifications. In this way, it strives to bridge the chasm between exemplar-based concrete thinking and generalized abstractions that are central to mathematically valid proof production.

THV draws on the power of collaboration to engage players in the formation of distributed knowledge. Collaborators are shown to do far more than share knowledge verbally and symbolically. In addition, people come to recognize the representational power and expressiveness of both their own bodies and that of their collaborators. These interactions require establishing a shared understanding as well as offering ways to extend their own embodied thoughts onto the world around them. THV also explores the power of collaborative co-creation of new mathematical content to share with others. Creators come to recognize some of the ways their novel mathematical ideas become grounded in movement. They can then use movement as a means for transmitting generalizable concepts to others, who demonstrate them through their own movements. By recognizing transfer as an inherently embodied process, THV offers alternative ways to build shared knowledge. This raises considerations about the role of the body in assessment, as learners may exhibit understanding through movements before they are aware of new emergent meanings or before they can verbalize new ways of thinking (Church & Goldin-Meadow, 1986).

Here, we have looked at the close relationship among empirical evidence, theory building, and design to investigate pathways for progress on complex issues facing GEL. We believe this work starts to establish an empirical basis for theorizing how linguistic and bodily knowledge among individuals and groups serve advanced mathematical reasoning, and the technological designs to support these types of social and embodied interactions. Still, this work has several limitations. One is the provisional nature of many of these empirical findings, which must face replication using larger and more varied samples, conducted with less direct input from the original research team. Another is that the principles themselves need to be tested with other digital platforms and using a broad array of student outcome measures before they can be really regarded as well established and generalizable. Our interest is to foster this additional research and the scrutiny it offers for theory and design. We take these initial steps as promising forays into a new class of theories and technologies for grounded and embodied learning.

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Figure 1

Participants reasoning about a conjecture. (a) Left*. A dynamic depictive gesture simulates transformations to reveal generalizable properties. (b)* Right*. Non-dynamic gesture of static properties of a simulated triangle.*

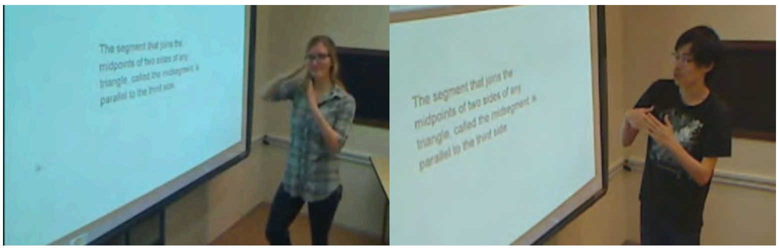

Figure 2
Flow of The Hidden Village. Bottom row from left: *(a) Meet a villager, (b) mimic directed actions of avatar, (c) free-responses in speech and gesture for a mathematical conjecture, (d) select a multiple-choice response, and (e) receive a knowledge token and expose the village map.* Top row: *Game play in a high school classroom.*

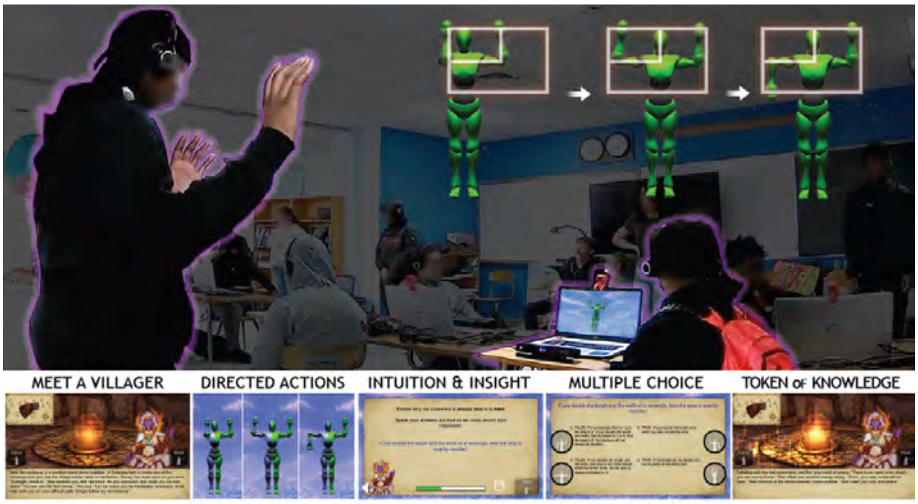

Figure 3
Example collaborative gestures (1) echoing, (2) mirroring, (3) joint gesture.

Figure 4
Left. *THV Conjecture Editor to create new conjecture for future game play.* Middle. *THV Pose Editor to create directed actions.* Right. *A new directed action sequence.*