Mathematical Knowledge is Embodied: Synergistic Contributions of Gesture and Speech During Geometry Proof Production

Kelsey E. Schenck, University of Wisconsin-Madison
Michael I. Swart, University of Wisconsin-Madison
Joseph E. Michaelis, University of Wisconsin-Madison
Mitchell J. Nathan, University of Wisconsin-Madison
Candace Walkington, Southern Methodist University

Abstract: The current study provides evidence that mathematical knowledge is embodied. Gesture and speech each make unique contributions to assessing students' (N=90) mathematical knowledge. Findings are consistent with embodiment theory, which posits that situation models of language are cognitive simulations of the situated problem space. In effect, reasoning within and about these simulations for generating transformational proofs is most effective when both verbal elements (i.e., signaling a situated model and generalizability) and non-verbal elements (i.e., representational and dynamic gestures) are integrated in the proof process. The analysis reveals that assessment of math reasoning improves when attending to both verbal and embodied aspects of students' communication. Student learning may benefit when educators operate with an embodied view of student knowledge.

Keywords: Embodied Cognition, Gesture, Geometric Proofs

Objective

How is advanced mathematical thinking embodied? Research in embodiment suggests that gestures impact reasoning and communicating about math (Alibali & Nathan, 2012; De Freitas & Sinclair, 2014; Hall & Nemirovsky, 2012). Gesture scholars, operating from cognitive, developmental, linguistic, and phenomenological perspectives, conceptualize gesture and speech as distinct, but co-constitutive and complementary in meaning (e.g., Goldin-Meadow, 2005; Hall & Nemirovsky, 2012; McNeill, 1992; Radford, 2009). Thus, speech and gesture may each provide unique and complementary information about what a person knows when they produce a mathematical proof (CadwalladerOlsker, 2011; Sweeney & Rasmussen, 2014; Marghetis, Edwards, and Núñez, 2014). This is especially likely for transformational proofs (Harel & Sowder, 2007), in contrast to authoritarian proofs ("the textbook says so") and perceptual proofs ("they look the same"). Transformational proofs are a class of deductive proofs that depend on the transformation of mathematical objects to establish three essential qualities: Generality, logical inference, and operational thought. Though there are several studies of experts' gestures during proof, few compare experts with non-experts. The objective of this study is to document the complementary role of gesture and speech by examining the unique contributions of each as predictors of expert and non-expert transformational proofs for conjectures in geometry. We conducted automated transcription analysis of participants' spoken proofs to document the forms of speech associated with transformational proofs and dynamic gesture production.

Theoretical Framework

The Embodied Nature of Mental Models and Mental Simulations

During problem solving, people often spontaneously construct a functional, dynamic mental image, or *situation model*, of the problem space – that typically includes the objects, events and relations within the given problem – by integrating information read from text with the readers' prior knowledge and personal experiences (Gentner & Stevens, 1983; Johnson-Laird, 1980; van Dijk & Kintsch, 1983). According to embodied cognition theory, our construction of these situation models is rooted in the ways our bodies function and interact with the world; reasoning about these models is based on mental simulations of our actions and sensations (both actually and metaphorically) of interactions with these physical and mental objects or entities (Barsalou, 1999; Glenberg, 1999). Therefore, a situation model is constructed and manipulated in part through the physical gestures we produce as simulated actions on the objects, space, or concepts within the problem situation (Hostetter and Alibali, 2008). When agents' bodily actions are compatible with the dynamic and relational qualities of a concept, certain cognitive benefits are observed (Lindgren & Johnson-Glenberg, 2013). For example, participants who first performed actions relevant to the enactment of core mathematical relationships of a given mathematical conjecture produced superior transformational proofs (Nathan et al., 2016). Text comprehension increases when readers use gestures that align with the valence of the meaning of the text (Chen & Bargh, 1999), while restricting gesture production can impair situation model formation (Nathan & Martinez, 2015).

The Embodied Nature of Mathematical Reasoning

What can we learn from a person's speech and gestures as they reasoning about the truth of a mathematical conjecture? Certainly, we can classify the parts of speech that are used (e.g., nouns, verbs, adjective, and pronouns). And, as with the situation model of a narrative text, people form situation models of mathematical information that helps them with their analytic reasoning (Mayer & Hegarty, 1996; Nathan, Kintsch, & Young, 1992). We can examine the contents of speech for textual indicators of the level of cohesion between the ideas and actions. (McNamara, Graesser, McCarthy & Cai, 2014; Zwann, 2015).

The record of people's co-speech gestures also reveals information about how they reason. Representational gestures such as pointing and tracing show the objects that are the focus on attention and their basic properties (Edwards, MooreRusso & Ferrara, 2014). In order to infer *generalizable properties*, people use situation models to simulate the dynamic properties of space and shape. This can be revealed by their use of dynamic gestures, which enact operations on mathematical objects (Garcia & Infante, 2012), and are closely linked to generating valid transformational proofs (Williams-Pierce et al., 2017; also see Figure 1).

Guided by a situation model-based framework for embodied mathematical reasoning, we analyzed the speech and gestures from videos of mathematical experts' and on-experts' reasoning about the validity of geometric conjectures. We set out to investigate the following two research questions: (1) Are speech and gesture each independently predictive of producing valid mathematical proofs? (2) What types of speech are most closely associated with dynamic gesture use?

Method

We chose to explore these research questions in the context of formulating mathematical proofs about geometric conjectures because proofs are highly conceptual (rather than procedural) and involve expressing logical chains of inference intended to make generalizable statements. In previous work (Authors, 2018), we found that use of dynamic gestures, along with expertise and spatial scores, were predictors of producing a valid transformational proof. Here, we extend that analysis to include additional data about the content of each participant's verbal reasoning, using Coh-Metrix, a validated text-mining tool that produces measures of several cognitive and linguistic indices, including situation model, connectives, lexical diversity, and syntactic complexity (McNamara et al., 2014; www.cohmetrix.com). In turn, we used the Coh-Metrix variables to analyze the association between verbal content and dynamic gesture production.

In the current study, students from a large midwestern university were identified as either math experts (advanced undergraduate math majors; n = 46) or non-experts (non-STEM pre-service teachers; n = 44). Each participant was shown a series of four mathematical conjectures (see Table 1) and asked to state whether they believed the conjecture to be always true or false and provide their reasoning. For their reasoning to be scored as a valid proof it had to meet three criteria: (1) logical (an inductive/deductive chain of reasoning), (2) operational (a cohesive goal-directed progression of operations through a chain of sub-goals), and (3) generalizable (establish universal relationships for an entire class of mathematical objects). Students' responses were video recorded and coded for proof validity and use of dynamic gestures (reliability κ = .903) and transcriptions were then analyzed using Coh-Metrix text analysis. We also collected demographic information, prior geometry knowledge, and measures of spatial reasoning (Ekstrom, French, Harman, & Derman, 1976) and phonemic fluency (desRosiers & Kavanagh, 1987).

Results and Discussion

In our analysis, we used the *lmer* R package to create mixed-effects logistic regression models to predict: valid transformational proofs, and the production of dynamic gestures, with participant ID and conjecture as random effects. For the transformational proof model, we included expertise, spatial scores, phonemic fluency scores, ethnicity, prior geometry knowledge, and use of dynamic gestures as fixed effects based on a previous analysis of this data (Authors, 2018). For all models, we added 21 variables, previously found to be significant predictors of gesture and valid proofs, provided by the Coh-Metrix analysis as fixed effects in our models (Table 2). All fixed effect variables were scaled (0-1). We then fit the models using a stepwise backwards elimination method (Hosmer & Lemeshow, 2000). We determined best fit model selection using ANOVA comparisons between models to test for significant reductions in deviance using a chi-square distribution, and calculated effect sizes for regression coefficients (Chinn, 2000).

Proofs

In our previous work, without contributions of parts of speech or situation model cohesion, we found that use of dynamic gestures (d = 1.44, p <.001), expertise (d = 0.69, p =.042), spatial scores (d = 0.16, p <.001) were significant positive predictors of valid proofs (Model 1, Table 3). A new analysis added language variables to the model, obtained from the text analysis. Three speech variables were associated with increased likelihood of generating a valid proof and made a significant reduction in model deviance (Model 2, Table 3): (1) a high proportion of intentional connective statements in their verbal reasoning, which indicates a more intentionally cohesive situational model (d = 0.29, p =.004); (2) increased verb use (d =

0.25, p =.037), which indicates action-oriented accounts; and (3) reduction of first person singular pronoun use (d = -0.43, p = 0.001), which signals more attention to non-self references such as discussing math objects. In this model, the use of dynamic gestures (d = 1.94, $p < .001$) remained a significant contributor.

As evidence for this model, we present the verbal content of a valid proof for a non-expert, female, who did not use any dynamic gestures in response to the circumscription conjecture (*see* Table 1 #4): "I believe that's true, because a circle has no, um, limit on how big the radius can be, *and so, for,* like, the length of the triangle, would just be the circumference that it's circled from. *So*, um, I believe that's true about most triangles." In this case, the participant showed an increased use of connective statements (in italics) in lieu of dynamic gestures.

Gestures and Speech

The best fit regression model revealed that dynamic gesture production (Table 4) was predicted based on increased use of comparing and contrasting connectives (d = 1.15, p = 0.017), increased use of verbs (d = 1.60, p = 0.001) and use second person pronouns (d = 1.52, $p < 0.001$), along with reduced usage of first person singular pronouns (d = -1.20, p = 0.017).

As evidence for this model, we present the speech and gestures for one of our participants (Figure 1, left) who used only second person pronouns and included comparing/contrasting connectives (both denoted in italics), while producing a dynamic gesture (occurring during bolded speech). While reasoning about similar triangles in conjecture 3 (*see* Table 1), she says: "*You* can have different lengths, like it could be a bigger or a smaller triangle *that still has all the same* angles. . . **So, if all, if the smallest triangle *with all the same angles* and *you* enlarged, if *you* made the triangle bigger, the angles wouldn't change, just the lengths of the sides** would change." The use of comparative connectives such as "with all the same angles" allows the participant to establish the similarities between the triangles while using dynamic gestures to manipulate the size of the triangle. The use of second person pronouns may indicate an external, more objective focus that can foster greater abstraction.

Conclusion

In this study we found evidence that speech patterns and dynamic gestures are related to the production of valid transformational proofs in mathematics (RQ1), and identifies the types of speech patterns most strongly related to dynamic gesture production (RQ2). Dynamic gestures were most evident when participants described contrasting relations that supported generalization and an audience perspective.

Students were more likely to produce valid proofs when their verbal reasoning *constructed and described the situation* (as indicated by intentional connectives), *simulated situated actions through verb use*, and objectified their references, as indicated by an *other-oriented (second-person) account* of their reasoning. Dynamic gestures provided independent support for the operational nature of proof construction not otherwise accounted for by spoken language factors. The production of dynamic gestures was related to the verbalization of contrasts and comparisons between features of mathematical objects.

In sum, this study provides evidence that gesture and speech each provide unique contributions to mathematical proof production. These findings are consistent with embodiment theory, which suggests that situation models of language are cognitive simulations of the situation (Glenberg, 1999). Limitations of this study highlight the need for additional research: First, a micro-scale analysis of the specific relationships between dynamic gesture use and situation based mathematical reasoning is needed to evaluate the nuances of embodied, model-based reasoning. Second, the correlational nature of this data calls for an intervention that can establish causal links between relevant action, situation modeling and proof performance.

Significance

Scientifically, our work supports and expands embodied accounts of mathematical reasoning by demonstrating that students' simulations for transformational proofs are most effective when integrating both verbal and gesture-based elements (Nathan et al., 2014). Educationally, our findings suggest accurate *assessment* of mathematical reasoning requires attending to both speech and gesture. Educators who implement this multimodal method of assessment may benefit from an expanded perception of their students' mathematical reasoning.

References

Alibali, M. W., & Nathan, M. J. (2012). Embodiment in mathematics teaching and learning: Evidence from learners' and teachers' gestures. *Journal of the Learning Sciences*, *21*(2), 247-286.

CadwalladerOlsker, T. (2011). What do we mean by mathematical proof? *Journal of Humanistic Mathematics, 1*(1), 33–60.

Chen, M., & Bargh, J. A. (1999). Consequences of automatic evaluation: Immediate behavioral predispositions to approach or avoid the stimulus. *Personality and social psychology bulletin*, *25*(2), 215-224.

Church, R. B., & Goldin-Meadow, S. (1986). The mismatch between gesture and speech as an index of transitional knowledge. *Cognition*, *23*(1), 43-71.

De Freitas, E., & Sinclair, N. (2014). *Mathematics and the body: Material entanglements in the classroom*. Cambridge University Press.

desRosiers, G., & Kavanagh, D. (1987). Cognitive assessment in closed head injury: Stability, validity and parallel forms for two neuropsychological measures of recovery. *International Journal of Clinical Neuropsychology, 9,* 162–173.

Edwards, L. D., MooreRusso, D., & Ferrara, F. (Eds.). (2014). *Emerging perspectives on gesture and embodiment in mathematics*. IAP.

Ekstrom, R. B., French, J. W., Harman, H., & Derman, D. (1976). *Kit of Factor-Referenced Cognitive Tests* (rev. ed.). Princeton, NJ: Educational Testing Service.

Dijkstra, K., Kaschak, M. P., & Zwaan, R. A. (2007). Body posture facilitates retrieval of autobiographical memories. *Cognition*, *102*(1), 139-149.

Garcia, N., & Infante, N. E. (2012). Gestures as facilitators to proficient mental modelers. In L. R. Van Zoest, J.-J. Lo, & J. L. Kratky (Eds.), *Proceedings of the 34th Annual Meeting of the North American Chapter of the International Group for the*

Psychology of Mathematics Education (pp. 289-295). Kalamazoo, MI: Western Michigan University.
Gentner, D., & Stevens, A. L. (1983). *Mental Models*. Hillsdale, New Jersey: Erlbaum.
Glenberg, A. (1999). Why mental models must be embodied. *Advances in Psychology*, *128*, 77-90.
Goldin-Meadow, S. (2005). *Hearing gesture: How our hands help us think*. Harvard University Press.
Hall, R., & Nemirovsky, R. (2012). Introduction to the special issue: Modalities of body engagement in mathematical activity and learning. *Journal of the Learning Sciences*, *21*(2), 207-215.
Harel, G., & Sowder, L. (2007). Toward comprehensive perspectives on the learning and teaching of proof. In F. Lester (Ed.), *Second handbook of research on mathematics teaching and learning*, *2*, (pp. 805-842), National Council of Teachers of Mathematics.
Hosmer, D. W., & Lemeshow, S. (2000). *Applied Logistic Regression.* New York: Wiley.
Hostetter, A. B. & Alibali, M. W. (2008). Visible embodiment: Gestures as simulated action. *Psychonomic Bulletin & Review*, *15*(3), 495-515.
Lindgren, R., & Johnson-Glenberg, M. (2013). Emboldened by embodiment: Six precepts for research on embodied learning and mixed reality. *Educational Researcher*, *42*(8), 445-452.
Marghetis, T., Edwards, L. D., & Núñez, R. (2014). More than mere handwaving: Gesture and embodiment in expert mathematical proof. In L.D. Edwards, F. Ferrara, & D. Moore-Russo (Eds.), *Emerging perspectives on gesture and embodiment in mathematics* (pp. 227–246). Charlotte, NC: Information Age Publishing.
Mayer, R. E., & Hegarty, M. (1996). The process of understanding mathematical problems. *The nature of mathematical thinking*, 29-53.
McNamara, D. S., Graesser, A. C., McCarthy, P. M, & Cai, Z. (2014). *Automated evaluation of text and discourse with Coh-Metrix.* Cambridge University Press: Cambridge.
McNeill, D. (1992). *Hand and mind: What gestures reveal about thought*. University of Chicago press.
Nathan, M. J., Kintsch, W., & Young, E. (1992). A theory of algebra-word-problem comprehension and its implications for the design of learning environments. *Cognition and instruction*, *9*(4), 329-389.
Nathan, M. J., & Martinez, C. V. (2015). Gesture as model enactment: the role of gesture in mental model construction and inference making when learning from text. *Learning: Research & Practice*, *1*, 4-37.
Nathan, M. J., Walkington, C., Boncoddo, R., Pier, E., Williams, C. C., & Alibali, M. W. (2014). Actions speak louder with words: The roles of action and pedagogical language for grounding mathematical proof. *Learning and Instruction*, *33*, 182-193.
Radford, L. (2009). Why do gestures matter? Sensuous cognition and the palpability of mathematical meanings. *Educational Studies in Mathematics*, *70*(2), 111-126.
Sweeney, G. & Rasmussen, C. (2014). More than mere handwaving. In L. D., Edwards, D. MooreRusso, & F. Ferrara (Eds.). (2014). *Emerging perspectives on gesture and embodiment in mathematics*. IAP.
van Dijk, T. A. & Kintsch, W. (1983). *Strategies of discourse comprehension*. New York: Academic Press.
Williams-Pierce, C., Pier, E. L., Walkington, C., Boncoddo, R., Clinton, V., Alibali, M. W.,

& Nathan, M. J. (2017). What we say and how we do: Action, gesture, and language in proving. J*ournal for Research in Mathematics Education*, *48*(3), 248-260.
Zwann, R. A. (2015). Situation models, mental simulations, and abstract concepts in discourse comprehension. *Psychonomic Bulletin & Review*, *23,* 1028-1034.

Tables and Figures

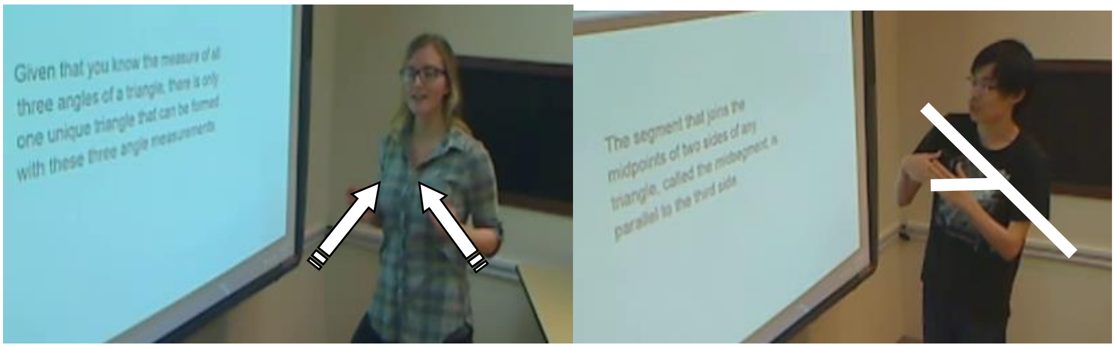

Figure 1. Participant (left) uses dynamic gesture to explore how changing the size of triangles does not change angle size, while a participant (right) uses a static gesture to explore the midsegment for one side of a triangle.

Table 1: The four mathematical conjectures used in the study, and their truth values

	Conjecture Text	Truth
1	The area of a parallelogram is the same as the area of a rectangle with the same length and height.	True
2	The segment that joins the midpoints of two sides of any triangle is parallel to the third side.	True
3	Given that you know the measure of all three angles of a triangle, there is only one unique triangle that can be formed with these three angle measurements	False
4	A circle can be circumscribed about any triangle.	True

Table 2

21 Coh-Metrix variables and their descriptions

Index	**Abbreviation**	**Description**
Descriptive		
	DESWC	Word count, number of words
Referential Cohesion		
	CRFNOa	Argument overlap- global overlap between sentence in terms of nouns
	CRFSOa	Stem overlap- global overlap between sentence in terms of nouns
Lexical Diversity		
	LDTTRc	Number of unique content words divided by the number of tokens of these words
	LDTTRa	Number of all unique words divided by the number of tokens of these words
Connectives		
	CNCALL	Incidence score of all connectives
	CNCADC	*Incidence score of adversative/contrastive connectives*
	CNCTemp	Incidence score of temporal connectives
	CNCTempx	Incidence score of extended temporal connectives
Situation Model		
	SMCAUSv	Incidence score of causal verbs
	SMINTEp	Incidence score of intentional actions, events, and particles (per 1000 words)
	SMCAUSr	Ratio of causal particles to causal verbs
	SMINTEr	*Ratio of intentional particles to intentional actions/events*
Syntactic Complexity		
	SYNLE	Mean number of worlds before the mian verb of the main clause in sentences
	SYNNP	Mean number of modifiers per noun-phrase
	SYNSTRUTt	Proportion of intersection tree nodes between all sentences and across paragraphs
Word Information		
	WRDVERB	*Incidence score of verbs*
	WRDPRO	Number of personal pronouns per 1000 words
	WRDPRP1s	*Incidence score of first person, singular pronouns*
	WRDPRP2	*Incidence score of second person pronouns*
	WRDFAMc	Rating of how familiar a word seems to an adult

Note. Italics indicate variable used in final model.

Table 3

Results of the Logistic Regression Predicting Transformational Proof

Variable	***β***	**SE**	***d***	**p-value**	
Model 1: Main Effects with Gesture					
(Intercept)	-3.91	1.70		.036	*
Expert	1.25	0.48	0.69	.042	*
Verbal	-0.02	0.02	-0.01	.938	
Spatial	0.29	0.09	0.16	.000	***
Ethnicity1 (White)	0.36	0.59	0.20	.846	
Ethnicity2 (Asian)	-0.23	0.60	-0.13	.912	
Geometry Knowledge	0.09	2.09	0.05	.969	
Operative Action	2.60	0.39	1.44	.000	***
Model 2: Main Effects with Gesture and Speech					
(Intercept)	-3.57	2.29		.118	
Expert	0.60	0.63	0.33	.336	
Verbal	0.00	0.02	0.00	.998	
Spatial	0.20	0.11	0.11	.080	
Ethnicity1 (White)	0.09	0.76	0.05	.901	
Ethnicity2 (Asian)	0.32	0.80	0.18	.688	
Geometry Knowledge	-1.73	2.82	-0.96	.539	
Operative Action	3.51	0.42	1.94	.000	***
Situation Model- Intentional Cohesion	0.52	0.18	0.29	.004	**
Verbs	0.45	0.21	0.25	.037	*
1st Person Pronouns	-0.77	0.24	-0.43	.001	**

Table 4

Results of the Logistic Regression Predicting Operative Action

Variable	***β***	**SE**	***d***	**p-value**	
(Intercept)	-5.32	2.07		.0378	*
Expert	0.86	0.50	0.47	.0531	
Spatial	0.27	0.10	0.15	.0432	*
Age	0.12	0.10	0.06	.1242	
Gender (Male)	4.10	2.72	2.27	.3871	
Comparative Connectives	0.21	0.15	1.15	.0172	*
Verbs	0.34	0.17	1.60	.0011	**
2nd Person Pronouns	0.50	0.16	1.52	.0009	***
1st Person Pronouns	0.01	0.17	-1,20	.0169	*