# Grounded and embodied mathematical cognition for intuition and proof playing a motion-capture video game

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**Abstract:** Proof, though central to mathematical practice, is rarely explored through the lens of embodiment because of the centrality of abstraction and generalization. We use the case of a high school geometry student to investigate two research questions: (1) How do embodied processes facilitate mathematical learning? (2) How can generalized mathematical truths be manifest through embodied processes that are grounded in particular movements? To engage the body, researchers developed a motion-capture video game, *The Hidden Village*, designed to elicit physical movement via in-game directed actions. In-game interactions complement logic and verbal forms of reasoning by promoting simulated actions, which are shown to recur during the student's proof and justification. Embodied theories of learning offer insights into how learners recruit body-based resources to foster meaning-making and generalization. Embodiment also offers new insights into recharacterizing mathematics curricula in terms of movement, and the promise of new forms of embodied learning technologies.

## Overview

Investigations of mathematical learning and reasoning are challenged to provide accounts of how students can express their understanding of abstract entities and relations and discover generalizations. Classical accounts such as Information Processing models of cognition (Miller, 2003) start with internal mental symbol structures that can be manipulated according to syntactic rules that are intended to "represent" cognitive processes even as they are unconnected to specific events or sensory experiences. As unconnected, amodal symbols, they provide a relatively direct account of generalization: Symbols as cognitive models can represent operations on individual instances (i.e., tokens) as well as classes of instances (i.e., types), as observed by C. S. Peirce (Wetzel, 1998). However, enticing the framework may be for describing general computational operations, Information Processing accounts of cognition suffer from the *symbol grounding problem* (Harnad, 1990; Searle, 1980) and so are poor at explaining how meaning arises. Embodied accounts of mathematical cognition offer an alternative account of meaning by *grounding* mathematical ideas, procedures, and signs to the actions and perceptions commensurate with their affordances (Gibson, 1977). Externalized symbols such as numerals can be spatially interpreted as placement on a number line; algebraic expressions as composite objects that can be decomposed and manipulated; and shapes as objects that can be transformed through translation, rotation, reflection and dilation (Lakoff & Núñez, 2000).

Mathematics deals with more than identifying and manipulating idealized entities such as numbers and shapes. Mathematics also deals with identifying generalized properties and relations that are always true – the notion of *proof*. Proof is defined by the mathematicians Guershon Harel and Larry Sowder (1998) as "the process employed by an individual to remove or create doubts about the truth of an observation" (p. 241). Proof is one of the central methods for engaging in authentic mathematical practice. Researchers have noted that the proof practices exhibited by math experts reveal embodied processes (Marghetis & Núñez, 2013). Student novices struggle to generate mathematically valid proofs (Bieda, 2010; Stylianides & Stylianides, 2017). However, little is known about the ways students embody mathematical proof. We investigate this topic with data from a student engaging with a mathematical conjecture from Euclid's *Elements* (Heath, 1956). Our investigation is in the context of interactions with a video game designed to foster grounded and embodied mathematical cognition (Barsalou, 2008; Shapiro, 2010). Design principles of the game posit that (1) engaging in mathematically relevant body movements can foster advanced mathematical reasoning; and (2) one's reasoning will be exhibited multimodally via movement and language that reveal both nonverbal and verbal forms of mathematical knowing. In addition to furthering our understanding of how learners embody mathematical proof, this case analysis provides an occasion to reflect on how a universal truth can be manifest through embodied processes that are grounded in particular movements. This investigation offers insights into how advanced forms of mathematical learning are grounded in embodied processes that support discovery and proof production, and point to the potential of movement-based learning environments for fostering meaningful, advanced mathematics education.

## Theoretical Background

For a skill as complex as geometric proof, explicit supports are needed for meaning making (Lehrer & Chazan, 1998). Evidence is mounting that sensorimotor activity can activate neural systems, which can, in turn, alter and induce cognitive states (Thomas, 2013) through the process of *action-cognition transduction* (Nathan, 2017; Nathan & Walkington, 2017). Recent innovations have explored ways that embodied design (Abrahamson, 2015) can inform a new class of educational technology interventions that activate users' perceptual-motor resources in service of mathematical learning. The current project relies on student use of a motion-capture video game, *The Hidden Village* (THV), designed to promote the grounded and embodied nature of geometric proof production (Walkington, Swart, Nathan, 2018). Central to the intervention is the game's use of *directed actions*—body movements that learners are directed to perform during gameplay — combined with language prompts to encourage player production of co-speech gestures that mediate their mathematical reasoning. The approach draws on the theory of *Gesture as Simulated Action* (GSA; Hostetter & Alibali, 2008, 2019) and an emerging literature on cognition and mathematics education showing that mathematical ideas can be learned through action-based interventions (Nathan, 2014). The result is externalized intellectual processes that highlight the embodied ways that students ground their mathematical reasoning in movement, such as depictive gestures (Alibali & Nathan, 2012) and operational language (Barsalou, 2008; Dehaene, 1997).

## Gestures, Directed Actions, and Mathematical Cognitions

Reviewing the literature on gesture, directed action, and learning reveals four empirically-based findings of note: (1) gesture production predicts learning and performance; (2) directed actions are a malleable factor for influencing cognition; (3) directed actions from earlier training opportunities leave a historical trace, or legacy, expressed through gestures during later performance; and (4) mathematics reasoning is further enhanced when actions are coupled with task-relevant speech, leading to coordinated action-speech events that support the reflection processes that are the hallmark of multimodal metacognition. Goldin-Meadow, Cook, & Mitchell (2009) proposed how student-generated speech mediates learning from actions: Carrying out specific actions directs learner attention to solution-relevant features of the task, which helps students confer meaning to their actions. Overall, these findings support the assertion that gestures produced during intellectual activity both convey and influence cognitive processes (Alibali & Nathan, 2012; Church & Goldin-Meadow, 1986; Fischer et al., 2015; Goldin-Meadow et al., 2009).

*Dynamic depictive gestures* (Garcia & Infante, 2012; Walkington et al., 2014) are important for the study of mathematics learning because they simulate transformations of an object through multiple states. This process facilitates discovery and generalization. **Figure 1** shows a dynamic depictive gesture that a student used to help prove the conjecture that "The sum of the lengths of any two sides of a triangle must be greater than the length of the remaining side." The gestures use embodied simulation of motion-based transformations to show that the two sides of the triangle whose combined lengths are less than the remaining side cannot meet.

Figure 1: A dynamic depictive gesture used in the process of proving the Triangle Inequality theorem.

Pier et al. (2014) demonstrated that the production of dynamic gestures contributes to predicting performance on mathematical proofs over and above the effects of variations in speech. Participants' (N = 120) speech patterns also independently predicted whether their justifications were mathematically sound. Pier et al. identified transformational speech patterns as particularly crucial to valid proofs. *Transformational speech* was defined as verbal descriptions of goal-directed manipulations of mathematical objects through conditional statements ("if... then...") and language that repeated key mathematical terms. They found that transformational speech and dynamic gesture independently accounted for statistically significant amounts of variance in a model predicting proof validity, suggesting embodied processes in both verbal and non-verbal form influence reasoning. This finding was replicated by mathematics experts and non-experts (N = 90) when justifying their mathematical reasoning for four geometry conjectures. Gesture production predicted proof performance ($d$ = 1.67), even when controlling for expertise, spatial reasoning, and occurrences of transformational speech acts (Nathan et al., 2018).

## Embodied Mathematics Through Video Game Play

Several learning sciences investigations examine embodied mathematics through technology and gameplay. Abrahamson (2012) explored how students' physical enactments of the covariation of two constant (but unequal) rates using hand motions detected by the Kinect platform fostered an understanding of proportional reasoning. This activity allowed students' unschooled, goal-directed sensorimotor activity to be structured through mediated activity to give rise to formal and generalizable mathematical ideas (Abrahamson, 2015). Similarly, Enyedy and Danish (2015) used a mixed-reality environment to support students' understanding of Newtonian force using motion-tracking technology. They found that verbal and physical reflection on embodied activities and first-person embodied play enabled students to engage deeply with challenging concepts. THV was designed to foster connections between movement, speech, and mathematical thought. Williams-Pierce (2016) showed embodied play fostered mathematical discovery of multiplicative relations. Results from intervention studies show THV gameplay helps learners generate mathematically valid proofs of the conjectures because mathematically relevant directed actions elicit dynamic gestures that enact simulated actions that support generalizable geometry reasoning (Walkington, Chelule, Woods, Nathan, 2018). Furthermore, performing *relevant* directed actions help learners significantly more than "irrelevant" directed actions (experimental controls that changed the action-conjecture pairing to make them irrelevant; Author, date).

This inquiry pursues a primary research question, (RQ1) How are embodied processes used to evaluate the veracity of a canonical geometric conjecture? Additionally, the investigation sought a deeper understanding of the power and limits of embodied proof by exploring a second question, (RQ2) How can a universal truth manifest through embodied processes that are fundamentally grounded in particular movements? Our case study analysis reveals important insights into these two questions.


Figure 2: Game flow of *The Hidden Village* (THV): (1) An opening sequence introduces players to the hidden village and calibrates the motion capture before play, (2) Each member of THV engages players who request physical participation (i.e., embodied) (3) Players move through a series of poses (i.e., *directed actions*) that turn green when successfully matched, (4) The villager requests players speak their answers (recorded via wireless headset and HD video)

## Methods

### Participant & Procedure

We selected an excerpt from a student from a Title 1 high school in the Midwestern United States, where 59% of the students receive free or reduced-priced lunch. We chose to do a detailed analysis of one student in order to show "signs of learning" (Kress & Selander, 2012) as it occurs in an area of complex mathematical thinking. We selected this specific participant because his video transcript was especially clear for illustrating the dynamic process of knowledge elaboration via embodied simulation. He was recruited to play THV (**Figure 2**), a video game that utilizes motion-capture technology to deliver an interactive math geometry curriculum based on theories of grounded and embodied mathematical cognition. In the game narrative, the player has been separated from a group of hikers and meets members of The Hidden Village who speak an unfamiliar language. Instead, the player is invited to participate in various activities such as weaving and dancing, which are, in actuality, mathematically relevant *directed actions* derived from prior observations of the gestures of people's successful proof production. The player's movements are processed in real-time using the MS Kinect 2 sensor array. Successfully matching in-game movements exposes the next challenge in the form of a mathematical conjecture (a statement that is either false or always true). The player decides its truth and is prompted for an explanation (that usually includes their producing co-speech gestures). A player then selects multiple-choice answers about the validity of the conjecture, and following each conjecture, the player acquires village "glyphs," and more of the village map is revealed (in-game achievements and progress), ultimately leading the player out of the village and back with the group.

### Coding

All gameplay was audio and video recorded. Researchers created transcripts of speech and gestures using Transana 3.21 and coded the correctness and types of proofs (following the proof taxonomy of Harel & Sowder, 2007). For a proof to be mathematically valid, it must independently demonstrate three criteria (Harel & Sowder, 2005): (1) generalizable across the entire class of mathematical entities, (2) logical inference, progressing through a deductive structure of goals and subgoals to formulate an argument, and (3) exhibit operational thought by applying valid, goal-directed operations on mathematical objects and observing their results

## Results

This case illustrates how actions invoked during gameplay foster mathematical reasoning involved in proof production. In prior work (e.g., Nathan & Walkington, 2017), we learned that participants were more successful at proof when they produced dynamic gestures, and we also learned that they were more likely to produce dynamic gestures when these were elicited through gameplay. This analysis focuses on the student's embodied mathematical reasoning after matching the directed actions prompted by the game, as shown in **Figure 2**, Phase 3 of the game cycle, which directed the student to cross his arms at one angle and then a second angle. The game then presented (**Figure 2**, Phases 4 & 5) the vertical angle theorem, "The opposite angles of two lines that cross are always the same," a variant of Euclid's Proposition 15. It did not refer to the relevance of the directed actions to the conjecture.

***Transcript #1***: [1] The statement is true. [2] Because the lines must always be lines like they can. [3] Assuming the lines are straight, and not-, and we're operating on the plane (0.5 sec), they um, (5.0 sec) [4] lines are, um (5.5 sec) ahh. [5] If one angle, uh, one of the opposite angles was different than, uh, the other opposite angle, then the lines would not be straight, [6] they would be (0.5 sec) uh, there would be um an angle at the intersecting point in one of the lines.

The student's initial response (**Figure 3**) can be considered in terms of 3 stanzas.

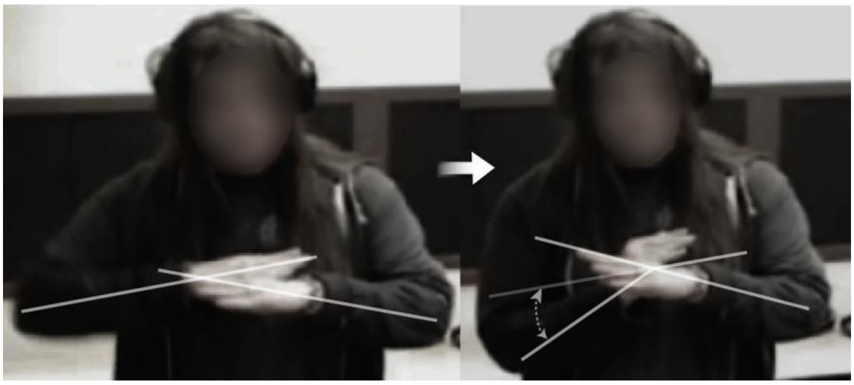

Figure 3: Student's initial Proof-By-Contradiction after performing in-game directed actions.

The first stanza (Transcript #1, Lines 1-2) reflects his snap judgment or intuition (Line 1) that the conjecture is true. He then adds (Line 2) "because the lines must always be lines…" In the second stanza (Lines 3-4), the student uses his arms to set up the spatial arrangement of lines described in the conjecture. As he says, "… operating on the plane," he crosses his right arm in front of his left (just as was done in the game), with fingers and palms flat and in line with his arms. He holds this position until he says (Line 4) "lines are, um" and trails off. During this utterance, he gently uncrosses his arms briefly and then crosses them again and holds them in place for 5.5 sec of silence.

The third stanza (Lines 5-6) exhibits a discovery phase similar to what Harel and Sowder (2007) describe as *ascertaining*, or the process of convincing oneself. His gaze is often on his arms as though studying them. Notably, he switches his arrangement, crossing the left arm in front of the right as he says (Line 5) "If one angle…" and while speaking, he pivots his right arm by lowering and raising the right elbow (maintaining his left arm position), which changes the angle at which the two arms cross – a *dynamic gesture*. He then makes a second dynamic gesture, bending all the fingers of his right hand at the palm, and states (Line 5) "then the lines would not be straight." He then uncrosses his arms and holds them both out in front, moving them in the rhythm of his speech (characteristic of beat gestures) in an apparent shift from ascertaining for himself, to a mode of *persuading* those who observe to him. During this, he says (Line 6) "they would be, uh, there would be um an angle at the [drops arms to his side] intersecting point …" He then crosses his arms again as he says, "in one of the lines."

Upon analysis, we see the student crossing his arms to make the two lines mentioned in the conjecture, in ways that match the directed actions during gameplay (Stanza 1). As noted, this sets up the stated conjecture and provides some cognitive offloading – a common benefit of embodied forms of reasoning that reduces the cognitive resources that would be expended tracking the location of the lines as they cross (Wilson, 2002). In an apparent effort to ascertain for himself the truth of the conjecture (Stanza 2), he bends his right hand (Line 3; also **Figure 3**) where the fingers meet the palm to show the possibility that the "lines" would not be straight if the two opposing vertical angles are not equal. This gestural explanation is a form of *Proof by Contradiction*. This approach is notable since it is first necessary to restate the conjecture in its contrapositive form (Line 5), by negating the consequent, "If one angle, uh, one of the opposite angles was different than, uh, the other opposite angle." This statement implies that the result is a negation of the antecedent, that the lines must not be straight, which he demonstrates with his bent hand. Proof by Contradiction also offers a powerful way to establish a universal truth by using specific instances of the arrangement of the lines (where one of the lines must be bent).

***Transcript #2***: [7] This statement is true, [8] because if two lines cross at any angle both lines must continue going in the direction they are going beforehand, like before the point at which they cross, um, and cannot change, [9] um therefore, since both, since a line always adds up to 180, a line always is 180 degrees from itself at any point, the divided area must add up to 180 degrees on both sides, [10] since the angles are both moving, since the lines are both moving, like they are not changing at the inflection point, this must be true.

THV then scaffolds the student's thinking by providing a hint that the directed actions cued by the characters in the game and performed by the player are relevant to the villagers. He still regards the conjecture as true (Line 7; **Figure 4**). When prompted to explain a second time, the player's movements are more transformational. He performs a dynamic gesture that enacts the pivoting movement of the lines at the "point at which they cross" (Line 8) but also enacts the continuity of the lines, reframing lines from the earlier static metaphor of Line-As-A-Set-of-Points (Lines 2-3) to be a Naturally-Continuous-Line (i.e., the path traced by a moving point; Lines 8 & 9), since lines "must continue going in the direction they are going beforehand."

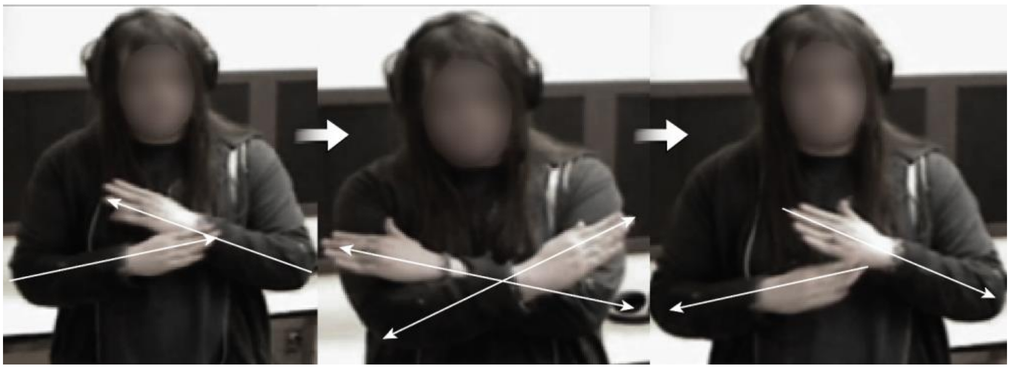

Figure 4: Student's Transformational Deductive Proof after receiving a hint to the relevance of the in-game directed actions. Lines are now simulated as naturally continuous movements in a direction.

This change showed he was focusing on the generalization in the proof for all lines, not just a particular example. Thus, his proof type changes from a Proof by Contradiction to a Transformational Deductive Proof that simulates the generalized properties of the opposing angles of all lines that cross (see **Figure 4**).

The student uses several of the arguments used by Euclid in proving Proposition 15, particularly that adjacent angles have measures that add up to 180° (Proposition 13); and "things which are equal to the same thing are also equal to one another" (Common Notion 1). By conceptualizing this in terms of lines that "move," the student notes that "both lines must continue going in the direction they are going beforehand," and actually improves on Euclid's published proof by establishing that no matter how one selects the intersecting lines, the proof holds.

## Discussion

We observed how a high school student embodied his ideas about Euclid's Proposition 15, often labeled the vertical angles theorem, by integrating the directed actions previously provided by a motion-capture game. The directed actions were used first to offload the cognitive demands of the spatial arrangement of the lines and ground these abstractions using his arms in the space in front of him. The student's proof process and his speech were accompanied by combinations of representational and dynamic gestures used to establish, identify and enact the movement of the lines, thereby demonstrating through motion-based simulation the generalized relationship between "any" intersecting lines and the angles they form. This case illustrates how embodied processes in the form of gestures and transformational speech are used to discover and evaluate the truth of a canonical geometric conjecture (RQ1). It also shows two ways that a universal truth can be established through particular embodied processes (RQ2): first, prior to the in-game hint, by restating the conjecture in terms that support Proof by Contradiction; and then, following a hint of the relevance of the in-game actions, as a transformational deductive proof that employed body-based simulation of "moving lines" along with transformational language of moving lines and angles to demonstrate that the conjecture must be true for all such intersecting lines.

Central to this proof production process are the interactions of the student as a player of an embodied video game designed to demonstrate the potential of grounded and embodied mathematical cognition as a platform for fostering embodied learning as exhibited in action- and body-based speech and gestures. Proof, though central to mathematical practice, is rarely explored through the lens of embodiment (Marghetis & Núñez, 2013). There is growing evidence that body-based interventions through instructional gestures and learners' movements can be effective for promoting mathematical reasoning (e.g., Gerofsky, 2018; Ng & Sinclair, 2015; Smith, King, & Hoyte, 2014; Valenzeno, Alibali & Klatzky, 2003). We offer two insights about the mathematical proof process. First, there is value in eliciting the body-based resources that children naturally have access to, but that connecting the potential relevance of these actions can further the utility of these resources during mathematical reasoning; and, second that there may be value in offering mathematical curricular activities that emphasize movement (e.g., transformational forms of proof) as a complement to logic-based forms that dominate traditional geometry classes.

This analysis is limited to the behavior of a single high school student reasoning about one conjecture. It documents how mathematical thinking can be studied and facilitated within a framework of embodiment. Our future work seeks to investigate the experimental effects that motion-based gameplay using directed actions can have on students' proof practices. While we have illustrated the use of this embodied learning intervention, there is nothing about this design that limits this to geometry, specifically, and we envision that this could be readily extended to proof in other areas of mathematics, such as number theory, graph theory, functions, probability and statistics, and calculus. Our objective is to further understand ways that body-based resources and language are used to foster mathematical learning and generalized reasoning, to expand the notion of what constitutes appropriate mathematical practices, and to offer evidence-based principles that inform the design of learning environments that embodied learners. One area of future research will be how these embodied forms of reasoning transfer outside of the game context. Drawing on a recent theory of embodied transfer, we expect that transfer to new contexts will occur as students engage processes of simulated action, such as repeating gesture movements internalized through gameplay (Nathan & Alibali, in press).

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